Efficacy of non-locality theorems "without inequalities" for pairs of spin-$\frac{1}{2}$ particles

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We argue that for a system of two spin-$\frac{1}{2}$ particles the recent theorems without inequalities, which show non-locality of quantum theory, fail in proving non-locality of any empirically valid theory sharing a set of correlations with quantum theory. In this case, a Greenberger-Horne-Shimony-Zeilinger argument cannot work.

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The subject of the present paper are the insights provided by the non-locality theorems without inequalities for two spin-$\frac{1}{2}$ particles. The theorem proposed by Hardy [1][2], in particular, has gained much interest in the literature on this subject [3]-[5]. Contrary to Bell's theorem [6], it does not make use of inequalities and works for almost all entangled states. In this letter we show that such new theorems, once proved that quantum theory is not a local and realistic theory, cannot extend this negative result to any theory which share only correlations, and not statistics with quantum theory. This stronger non-locality proof is attained by Greenberger, Horne, Shimony and Zeilinger (*GHSZ*, from now on) for a system consisting of at least three particles [7]. We show

that the method of GHSZ cannot apply in the case of pairs of two-level particles.

Now we present an equivalent reformulation of Hardy's argument. It involves two spin-$\frac{1}{2}$, space-like separated particles: particle 1 and particle 2. By $S_k(\mathbf{n})$ $(k = 1, 2)$ we denote the 1-0 observable which assumes value 1 (resp., 0) when the spin of particle k in direction $\mathbf{n}$ is $\frac{1}{2}\hbar$ (resp., $-\frac{1}{2}\hbar$). Four particular directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ are also considered, such that $\mathbf{n}_j$ and $\mathbf{n}_{j+2}$ are not parallel. The ingredients of Hardy's theorem are the following definition of element of reality and statements (i), (ii) and (iii).

DEFINITION. 1. – *We say that an observable S is an element of reality [equal to s] if a value [s] is assigned to S, albeit unknown, such that a measurement of S would yield that particular value [s].*

$i)$ *Principle of locality and reality*

Let S_1 and S_2 be two physical magnitudes which are measurable in two space-like separated regions. If a measurement of S_1 allows the prediction of the outcome of a measurement of S_2, then S_2 is an element of reality, *no matter whether S_1 is actually measured or not.*

$ii)$ *Three correlations*

1) $S_1(\mathbf{n}_1) \rightarrow S_2(\mathbf{n}_2)$; by this formula it is meant that if $S_1(\mathbf{n}_1)$ and $S_2(\mathbf{n}_2)$ are measured, then the outcome 1 for $S_1(\mathbf{n}_1)$ implies that also the outcome of $S_2(\mathbf{n}_2)$ is 1.

2) $S_2(\mathbf{n}_2) \rightarrow S_1(\mathbf{n}_3)$;

3) $S_1(\mathbf{n}_3) \rightarrow S_2(\mathbf{n}_4)$.

iii) *Quantum statistics.*

The probability of measuring the outcome 1 for the 1-0 observable represented by the projection operator $\hat{P}$ when the state vector is ψ (with $\|\psi\| = 1$), is given by

$$p_\psi(\hat{P}) = \langle \psi \mid \hat{P}\psi \rangle.$$

Let us suppose that a measurement of $S_1(\mathbf{n}_1)$ yields outcome 1. Then correlations (*ii*), together with principle (*i*), imply that $S_2(\mathbf{n}_2)$, and hence $S_1(\mathbf{n}_3)$ and $S_2(\mathbf{n}_4)$, are elements of reality equal to 1. Therefore conditions (*i*) and (*ii*), without using (*iii*), imply

$$S_1(\mathbf{n}_1) \;\rightarrow\; S_2(\mathbf{n}_4). \tag{1}$$

From the quantum theoretical point of view, condition (*ii*) forces the system in a precise quantum state ψ (see for instance [4]). Hence we can use such ψ to compute, by (*iii*), the quantum probability of obtaining outcomes 1 and 0 from a simultaneous measurement of $S_1(\mathbf{n}_1)$ and $S_2(\mathbf{n}_4)$, respectively. Such probability turns out to be different from 0 [1][2]. In other words, the quantum theoretical prediction contradicts (1). So we have the following logical situation.

$$\text{Hardy's Theorem} - \begin{cases} (i) \text{ and } (ii) \quad \Rightarrow \quad (1) & \text{(2i)} \\ (ii) \text{ and } (iii) \quad \Rightarrow \quad \text{not } (1) & \text{(2ii)} \end{cases}$$

A first important consequence of Hardy's theorem is that

I – *(ii) and (iii) are not consistent with (i), i.e. quantum theory is not "local and realistic" (in the sense that it does not satisfy (i)).*

No experiment is needed to get such a conclusion, but it is drawn on a purely theoretical basis. Another proof of this result has been recently

given by Stapp [8]; in such a proof Stapp reaches a contradiction by requiring the quantum correlations (ii) plus the quantum prediction *not (1)* (which is not a correlation), but only a *locality* condition, while the *reality* condition is derived by rigorously expliciting the *counterfactual* reasoning implicit in (i).

On the contrary, a second, stronger conclusion requires an experiment performed under experimental conditions which ensure that

(ec) *correlations* (ii) *hold according to quantum theory.*

It is fair enough that in such experiment a simultaneous measurement of $S_1(\mathbf{n}_1)$ and $S_2(\mathbf{n}_4)$ respectively yields outcomes 1 and 0 – just one time – (i.e., not (1)), to conclude by (2i) that

II – *every empirically valid theory in which correlations* (ii) *hold if they hold according to quantum theory, does not satisfy the principle of locality and reality* (i); therefore non-locality must be extended to any "realistic" theory which shares correlations (ii) with quantum theory.

We stress that in the latter argument the experimental test with result "not (1)" plays a necessary role. Of course, the occurrence of the experimental result "(1)" would falsify the theory, i.e. it would be *not empirically valid.* Moreover, II implies I.

The first aim of the present work is to show that Hardy's type argument, being successful with respect to conclusion I, fails in reaching II, because the *required experimental conditions (ec) are not realizable.* To explicitly see this, we consider the quantum theoretical description of the two space-like separated spins, in the Hilbert space $\mathbf{C}_1^2 \otimes \mathbf{C}_2^2$, where $\mathbf{C}_k^2$ is the Hilbert space for describing the spin of par-

ticle k. Let (u_k, v_k) be an orthonormal basis of $\mathbf{C}_k^2$. An orthonormal basis for the entire space is $(u_1 \otimes u_2, u_1 \otimes v_2, v_1 \otimes u_2, v_1 \otimes v_2)$. Since any projection operator of $\mathbf{C}_k^2$ may be written in the form $E_k^{\theta,\phi} = \begin{bmatrix} \cos^2\theta/2 & e^{-i\phi/2}\cos\theta/2\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2\sin\theta/2 & \sin^2\theta/2 \end{bmatrix}$ by a suitable choice of the angles θ and ϕ, then the projection operator $\hat{S}_1(\mathbf{n}) = E_1^{\theta,\phi} \otimes \mathbf{1}$ (resp. $\hat{S}_2(\mathbf{n}) = \mathbf{1} \otimes E_2^{\theta,\phi}$) of $\mathbf{C}_1^2 \otimes \mathbf{C}_2^2$ represents the observable $S_1(\mathbf{n})$ (resp. $S_2(\mathbf{n})$), where $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

If S_1 and S_2 are two measurable together 1-0 observables, according to quantum theory the correlation $S_1 \to S_2$ holds if and only if the quantum probability of measuring 1 for S_1 and 0 for S_2 is 0, i.e. if and only if

$$\langle \hat{S}_1(\mathbf{1} - \hat{S}_2)\psi \mid \psi \rangle = 0 \quad \text{iff} \quad \hat{S}_1\psi = \hat{S}_1\hat{S}_2\psi. \tag{3}$$

To emphasize the physical meaning of (3) we rewrite it in the following form, which explicitly exhibits the state dependence of the correlation.

$$\hat{S}_1 \overset{\psi}{\to} \hat{S}_2$$

Let $\hat{S}_1(\mathbf{n}) = F \otimes \mathbf{1}$ and $\hat{S}_2(\mathbf{m}) = \mathbf{1} \otimes A$ be two space-like separated 1-0 quantum observables. If we choose the basis of $\mathbf{C}_1^2$ as the basis of the eigenvectors of F, i.e. if $Fu_1 = u_1$ and $Fv_1 = 0$, then F is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\hat{S}_1(\mathbf{n}) = F \otimes \mathbf{1} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ represents the vector state ψ, the condition $\hat{S}_1(\mathbf{n}) \overset{\psi}{\to} \mathbf{1} \otimes A$ holds if and only if $A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$; therefore, when $\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (iff $\hat{S}_1(\mathbf{n})\psi \neq 0$), there is a unique projection operator A satisfying such condition, namely $A = \frac{1}{|a|^2+|b|^2}\left|\begin{bmatrix} a \\ b \end{bmatrix}\rangle\langle\begin{bmatrix} a \\ b \end{bmatrix}\right|$. Thus the following statement holds.

PROPOSITION 1. *Let ψ be any state vector of the entire system, and $\hat{S}_1(\mathbf{n}) = F \otimes \mathbf{1}$, where F is a projection operator of $\mathbf{C}_1^2$. Provided that $\hat{S}_1(\mathbf{n})\psi \neq 0$ there is a unique projecion operator $A = \frac{1}{|a|^2+|b|^2}\left|\left[\begin{array}{c} a \\ b \end{array}\right]\rangle\langle\left[\begin{array}{c} a \\ b \end{array}\right]\right|$ of $\mathbf{C}^2$ such that $\hat{S}_1(\mathbf{n}) \stackrel{\psi}{\to} \mathbf{1} \otimes A$.*

Proposition 1 implies that, once choosen the first direction $\mathbf{n}_1$ in such a way that $\hat{S}_1(\mathbf{n}_1)\psi \neq 0$, there is a unique triple of directions $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ such that conditions (ii) hold. An absolute absence of errors in the relative orientations $\mathbf{n}_1$ and $\mathbf{n}_4$ in a real experiment is impossible. Since the existence of an experimental error on $\mathbf{n}_1$ and $\mathbf{n}_4$, whatever be its entity, provokes the breadown of quantum correlations (ii), the possibility of an experimental test of (1) under conditon (ec) is completely hopeless. For these reasons we cannot reach conclusion (II) by using Hardy's theorem.

REMARK. Conclusion I, with the strict correlations of the singlet state instead of (ii), is provided also by Bell's theorem without the need of experiments. The necessity of the experiment rises only to get conclusion II. The experiment required by Bell's argument consists in measuring three alternative pairs of observables, whose *reality* is ensured *for all directions*, to check whether satisfies Bell's inequality [9]. Moreover, according to quantum theory, the statistical magnitudes involved in Bell's inequalities are *continuous* functions of the orientations of the measuring apparatuses. Therefore the experimental violation of the inequalities is expected to be guaranteed by limiting the experimental errors on such orientations within suitable bounds. By using Hardy's argument, the much simpler experiment consists in meausuring only the *two* observables $S_1(\mathbf{n}_1)$ and $S_2(\mathbf{n}_4)$ until the occurrence of the pair of outcomes (1,0) realizes the violation of (1); unfortunately, such ex-

periment is unrealizable. So, although Hardy's theorem "is the best version of Bell's theorem" because of its "highest attainable degree of simplicity and physical insight" [3], it suffers a lack of epistemological efficacy with repect the older Bell's argument.

On the contrary, the non-locality theorem without inequalities presented by GHSZ [7] reaches a conclusion like II without the need of experiments, but it requires at least three particles. Here we briefly sketch the argument in the case of four spin-$\frac{1}{2}$ space-like separated particles. The ingredients of GHSZ's theorem are

i) *Principle of locality and reality*

The same as for Hardy's theorem.

ii′) *Correlations*

A finite set of correlations, each correlation involving four pairwise space-like separated spin observables. One of the observables involved in these correlations is $S_1(\mathbf{n}_0)$, i.e. the 1-0 observable describing the spin of the first particle in direction $\mathbf{n}_0 = (1, 0, 0)$.

Notice that quantum statistics (iii) is not present in these premises. GHSZ proved that if conditons (i) and (ii') hold, then $S_1(\mathbf{n}_0)$ turns out to be an element of reality.

The central role in the argument is played by the following statement.

$$\text{GHSZ's T}\text{HEOREM.} - \begin{cases} (i) \text{ and } (ii') \;\Rightarrow\; S_1(\mathbf{n}_0) \to 1 - S_1(\mathbf{n}_0) & \text{(4i)} \\ (i) \text{ and } (ii') \;\Rightarrow\; 1 - S_1(\mathbf{n}_0) \to S_1(\mathbf{n}_0) & \text{(4ii)} \end{cases}$$

It must be said that (4i), by itself, does not necessarily yield inconsistency: it is a correlation stating that outcome 1 for $S_1(\mathbf{n}_0)$ cannot occur. But (4ii) says that outcome 0 is impossible too. As a conse-

quence, it turns out impossible to consistently assign a value to $S_1(\mathbf{n}_0)$, while it is an element of reality. Therefore, every theory which predicts correlations (ii') is inconsistent with the principle (i). Since there is a quantum state ψ for which correlations (ii') do hold, GHSZ's theorem implies conclusion II (with (ii') replacing (ii)), and no experiment is needed to get this result. Thus GHSZ prove the non-locality of any *realistic* theory which shares with quantum theory the set (ii') of correlations, without inequalities and without experiments. However, their proof holds for systems consisting of at least three particles.

GHSZ's argument suggests the idea of proving conclusion II for a system of *two* spin-$\frac{1}{2}$ particles, by following the same logical lines which avoid the necessity of (unrealizable) experiments. To realize such a program it is necessary, *at least*, to fulfil the following tasks (a) and (b).

a) *To find a set $\mathcal{R}$ of correlations, each correlation involving two space-like separated 1-0 observables such that from $\mathcal{R}$ and (i) derives the contradiction*

$$S_1(\mathbf{n}_1) \to 1 - S_1(\mathbf{n}_1) \tag{5}$$

where $S_1(\mathbf{n}_1)$ is one of the observables involved in correlations $\mathcal{R}$.

To establish the second task, we notice that according to quantum theory

$$\hat{S}_1(\mathbf{n}_1) \stackrel{\psi}{\to} \mathbf{1} - \hat{S}_1(\mathbf{n}_1) \quad \text{iff} \quad \hat{S}_1(\mathbf{n}_1)\psi = 0$$

and quantum theory by itself, i.e. without further assumptions as (i), is a *consistent theory*. Then, in order that (5) be a contradiction, it must be not predicted by quantum theory, i.e. we have to require $\hat{S}_1(\mathbf{n}_1)\psi \neq 0$. More generally, if $\hat{S}_1(\mathbf{n}_1)\psi = 0$ or $(\mathbf{1} - \hat{S}_1(\mathbf{n}_1))\psi = 0$,

then ψ is not an entangled state vector, i.e. it has the form $\varphi_1 \otimes \varphi_2$. Hence no quantum correlation holds between two space-like separated observables. In this case the locality and reality principle plays no role and therefore no contradiction can take place because of it. Thus, another indispensable task of a GHSZ-type program is

b) *To find a quantum state ψ such that*
   b.i) *correlations $\mathcal{R}$ hold according to quantum theory, and*
   b.ii) $\hat{S}_1(\mathbf{n}_1)\psi \neq 0 \neq (\mathbf{1} - \hat{S}_1(\mathbf{n}_1))\psi$.

In the remaining part of our work we show that these two tasks are unrealizable simultaneously.

First we introduce the concept of "chain" of correlations, elsewhere called *ladder* [5]. By *chain* we mean any finite, ordered sequence $\mathcal{C} = \{S_1(\mathbf{m}_1), S_2(\mathbf{m}_2), S_1(\mathbf{m}_3), ..., S_1(\mathbf{m}_{2k-1}), S_2(\mathbf{m}_{2k})...\}$ of "local" 1-0 observables such that the following chain of correlations holds.

$$S_1(\mathbf{m}_1) \to S_2(\mathbf{m}_2) \to \cdots \to S_2(\mathbf{m}_{2k}) \to S_1(\mathbf{m}_{2k+1}) \to \cdots \tag{6}$$

Lemma 1. – Let $\mathcal{C} = \{S_1(\mathbf{m}_1), S_2(\mathbf{m}_2), S_1(\mathbf{m}_3), ...\}$ be a chain. If the state vector ψ is such that (6) hold according to quantum theory, i.e. if

$$\hat{S}_i(\mathbf{m}_k) \overset{\psi}{\to} \hat{S}_{3-i}(\mathbf{m}_{k+1}), \quad \forall \hat{S}_i(\mathbf{m}_k), \hat{S}_{3-i}(\mathbf{m}_{k+1}) \in \mathcal{C}, \tag{7}$$

then

$$\hat{S}_1(\mathbf{m}_1)\psi \neq 0 \quad \Rightarrow \quad \hat{S}_i(\mathbf{m}_k)\psi \neq 0 \quad \forall \hat{S}_i(\mathbf{m}_k) \in \mathcal{C}. \tag{8}$$

Lemma 2. – $\hat{S}_1(\mathbf{n}) \overset{\psi}{\to} \hat{S}_2(\mathbf{m})$ iff $\mathbf{1} - \hat{S}_2(\mathbf{m}) \overset{\psi}{\to} \mathbf{1} - \hat{S}_1(\mathbf{n})$.

We do start our argument by considering *any* finite set $\mathcal{B} = \{S_r(\mathbf{n}_s)\}$ of "local" 1-0 observables, with $S_1(\mathbf{n}_1) \in \mathcal{B}$, endowed with a finite set

$\mathcal{R} = \{[S_1(\mathbf{n}_\lambda) \to S_2(\mathbf{n}_\rho)]\}$ of correlations. Then we show that there is no state vector ψ which satisfies (b.ii) such that

$$\begin{cases} [S_1(\mathbf{n}_\lambda) \to S_2(\mathbf{n}_\rho)] \in \mathcal{R} \quad \Rightarrow \quad \hat{S}_1(\mathbf{n}_\lambda) \xrightarrow{\psi} \hat{S}_2(\mathbf{n}_\rho) & \text{(b.i)} \\ \mathcal{R} \text{ and } (i) \quad \Rightarrow \quad S_1(\mathbf{n}_1) \to 1 - S_1(\mathbf{n}_1) & \text{(a)} \end{cases}$$

In so doing we do not lose generality. Indeed, *any correlation* between two measurable together 1-0 observables S_1 and S_2 consists in nothing else but the fact that some of the four pairs of outcomes $(0,0), (0,1), (1,0), (1,1)$ are impossible. For instance, suppose that (1,0) and (0,1) cannot occur. This particular correlation is expressed by the two formulas $S_1 \to S_2$ and $S_2 \to S_1$. We assume that the following *obvious* rule must hold in $\mathcal{R}$.

$$(R1) \quad [S_1(\mathbf{n}) \to S_2(\mathbf{m})] \in \mathcal{R} \quad \Leftrightarrow \quad [1 - S_2(\mathbf{m}) \to 1 - S_1(\mathbf{n})] \in \mathcal{R}.$$

According to (b) we have to assume $\hat{S}_1(\mathbf{n}_1)\psi \neq 0$ and $(\mathbf{1} - \hat{S}_1(\mathbf{n}_1))\psi \neq 0$. The two correlations $\hat{S}_i(\mathbf{n}_\lambda) \xrightarrow{\psi} \hat{S}_{3-i}(\mathbf{n}_\alpha)$ and $\hat{S}_i(\mathbf{n}_\lambda) \xrightarrow{\psi} \hat{S}_{3-i}(\mathbf{n}_\beta)$ imply $\hat{S}_{3-i}(\mathbf{n}_\alpha) = \hat{S}_{3-i}(\mathbf{n}_\beta)$ (by prop.1 and lemma 1). Therefore in $\mathcal{B}$ there is a unique maximal chain $\mathcal{C}_1$ containing $S_1(\mathbf{n}_1)$ and a unique maximal chain $\mathcal{C}_0$ containing $(1 - S_1(\mathbf{n}_1))$. This means that *there is no correlation in $\mathcal{R}$ between any observable in $\mathcal{C}_1$ or in $\mathcal{C}_0$ and any other observable in $\mathcal{B} \setminus (\mathcal{C}_1 \cup \mathcal{C}_0)$*. Therefore, the correlation $S_1(\mathbf{n}_1) \to 1 - S_1(\mathbf{n}_1)$ can be derived by using (i) only within the correlations in the chains $\mathcal{C}_1$ or $\mathcal{C}_0$.

Now, for *any* pair $(S_i(\mathbf{n}_k), S_j(\mathbf{n}_h))$ of measurable together observables in the same chain $\mathcal{C}$ (i.e. such that $[\hat{S}_i(\mathbf{n}_k), \hat{S}_j(\mathbf{n}_h)] = \mathbf{0}$), either $S_i(\mathbf{n}_k) \to S_j(\mathbf{n}_h)$ or $S_j(\mathbf{n}_h) \to S_i(\mathbf{n}_k)$ can be always derived by using principle (i) like we have done to get (1). Therefore, within a

chain $\mathcal{C}$ the only rule provided by (i) for deriving new correlations other than those provided by $\mathcal{R}$ is the following

$$(R2) \qquad S_i(\mathbf{n}_k) \rightarrow S_j(\mathbf{n}_h) \quad \text{iff} \quad k \leq h.$$

As a consequence, the correlation $S_1(\mathbf{n}_1) \rightarrow 1 - S_1(\mathbf{n}_1)$ can be derived from $\mathcal{R}$ and (i) only if in $\mathcal{C}_1$ the observable $S_1(\mathbf{n}_1)$ precedes $1 - S_1(\mathbf{n}_1)$. This may happen only if there exists $S_2(\mathbf{n}_{2k})$ in the chain $\mathcal{C}_1$ such that $S_2(\mathbf{n}_{2k}) \rightarrow 1 - S_1(\mathbf{n}_1)$. But the following proposition 2 states that such $S_2(\mathbf{n}_{2k})$ cannot exist whenever $S_1(\mathbf{n}_1)\psi \neq 0$.

Proposition 2. – Let $\mathcal{C} = \{S_1(\mathbf{n_1}), S_2(\mathbf{n}_2), S_1(\mathbf{n}_3), ...\}$ be a chain such that $S_2(\mathbf{n}_{2k}) \rightarrow 1 - S_1(\mathbf{n}_1)$ for some k. If there is $\psi \in \mathbf{C}^2 \otimes \mathbf{C}^2$ such that correlations (6) hold according to quantum theory, then $\hat{S}_1(\mathbf{n}_1)\psi = 0$.

Proof. Let ψ be a vector state such that (6) hold according to quantum theory. Let k be such that

$$\hat{S}_2(\mathbf{n}_{2k}) \stackrel{\psi}{\rightarrow} \mathbf{1} - \hat{S}_1(\mathbf{n}_1) \tag{9}$$

We prove that $\hat{S}_1(\mathbf{n}_1)\psi = 0$. Indeed, if $\hat{S}_1(\mathbf{n}_1)\psi \neq 0$, since $\hat{S}_2(\mathbf{n}_{2k}) \stackrel{\psi}{\rightarrow} \hat{S}_1(\mathbf{n}_{2k+1})$ and (9) hold, proposition 1 and lemma 1 imply $\mathbf{1} - \hat{S}_1(\mathbf{n}_1) = \hat{S}_1(\mathbf{n}_{2k+1})$, i.e.

$$\hat{S}_1(\mathbf{n}_1) = \mathbf{1} - \hat{S}_1(\mathbf{n}_{2k+1}). \tag{10}$$

If $\hat{S}_1(\mathbf{n}_1)\psi = \psi$, then from (10) we have $\hat{S}_1(\mathbf{n}_{2k+1})\psi = 0$, contrary to lemma 1. Then $0 \neq \hat{S}_1(\mathbf{n}_1)\psi \neq \psi$ and $0 \neq \mathbf{1} - \hat{S}_1(\mathbf{n}_1)\psi \neq \psi$ hold. More generally, using lemmas 1 and 2, it can be proved that

$$0 \neq \hat{S}_r(\mathbf{n}_s)\psi \neq \psi \quad \text{and} \quad 0 \neq (\mathbf{1} - \hat{S}_r(\mathbf{n}_s))\psi \neq \psi \tag{11}$$

hold for all $S_r(\mathbf{n}_s) \in \mathcal{C}$. Since

$$\begin{cases} \hat{S}_1(\mathbf{n}_1) \stackrel{\psi}{\rightarrow} \hat{S}_2(\mathbf{n}_2) \\ \mathbf{1} - \hat{S}_1(\mathbf{n}_{2k+1}) \stackrel{\psi}{\rightarrow} \mathbf{1} - \hat{S}_2(\mathbf{n}_{2k}), \end{cases}$$

(10) and prop.1 imply

$$\hat{S}_2(\mathbf{n}_2) = \mathbf{1} - \hat{S}_2(\mathbf{n}_{2k}).$$

By iterating this argument, making use of prop.1 and (11), we get

$$\hat{S}_{i(j)}(\mathbf{n}_{1+j}) = \mathbf{1} - \hat{S}_{i(j)}(\mathbf{n}_{2k+1-j}),$$

for all $j = 1, 2, ..., 2k$, where $i(j) = \frac{3-(-1)^j}{2} \in \{1, 2\}$ is the appropriate index. In particular, for $j = k$ we get the "impossible" equation

$$\hat{S}_{i(k)}(\mathbf{n}_{k+1}) = \mathbf{1} - \hat{S}_{i(k)}(\mathbf{n}_{k+1}). \tag{12}$$

Prop. 2 completes our argument against the possibility of proving non-locality of a realistic theory, describing two space-like separated two-level sub-systems, which shares a set of correlations with quantum theory, by using a GHSZ type – without inequalities and without experiment – method.

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